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Degree Formulae for Grassmann Bundles, II

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Abstract. Let X be a non-singular quasi-projective variety over a field, and let $\mathcal{E}$ be a vector bundle over X. Let $\mathbb{G}_X(d,\mathcal{E})$ be the Grassmann bundle of $\mathcal{E}$ over X parametrizing corank d subbundles of $\mathcal{E}$ with projection $\pi : \mathbb{G}_X(d,\mathcal{E}) \to X$, and let $\mathcal{Q} \leftarrow \pi^*\mathcal{E}$ be the universal quotient bundle of rank d. In this article, a closed formula for $\pi_* \operatorname{ch}(\det \mathcal{Q})$, the push-forward of the Chern character of the Plücker line bundle $\det \mathcal{Q}$ by π is given in terms of the Segre classes of $\mathcal{E}$. Our formula yields a degree formula for $\mathbb{G}_X(d,\mathcal{E})$ with respect to $\det \mathcal{Q}$ when X is projective and $\wedge^d\mathcal{E}$ is very ample. To prove the formula above, a push-forward formula in the Chow rings from a partial flag bundle of $\mathcal{E}$ to X is given.

0. Introduction

Let X be a non-singular quasi-projective variety of dimension n defined over a field of arbitrary characteristic, and let $\mathcal{E}$ be a vector bundle of rank r over X. Let $\mathbb{G}_X(d,\mathcal{E})$ be the Grassmann bundle of $\mathcal{E}$ over X parametrizing corank d subbundles of $\mathcal{E}$ with projection $\pi : \mathbb{G}_X(d,\mathcal{E}) \to X$, and let $\mathcal{Q} \leftarrow \pi^*\mathcal{E}$ be the universal quotient bundle of rank d on $\mathbb{G}_X(d,\mathcal{E})$. We denote by θ the first Chern class $c_1(\det \mathcal{Q}) = c_1(\mathcal{Q})$ of $\mathcal{Q}$, and call θ the *Plücker class* of $\mathbb{G}_X(d,\mathcal{E})$: In fact, the determinant bundle $\det \mathcal{Q}$ is isomorphic to the pull-back of the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\wedge^d\mathcal{E})}(1)$ of $\mathbb{P}_X(\wedge^d\mathcal{E})$ by the relative Plücker embedding over X.

The purpose of this article is to study the push-forward of powers of the Plücker class to X by π, namely, $\pi_*(\theta^N)$, where $\pi_* : A^{*+d(r-d)}(\mathbb{G}_X(d,\mathcal{E})) \to A^*(X)$ is the push-forward by π between the Chow rings. The main result is a closed formula for the push-forward of $\operatorname{ch}(\det \mathcal{Q}) := \exp\theta = \sum_{N\geq 0} \frac{1}{N!}\theta^N$, the Chern character of $\det \mathcal{Q}$ in terms of the Segre classes of $\mathcal{E}$, as follows:

Theorem 0.1. *We have*

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_k \frac{\prod_{0\leq i<j\leq d-1}(k_i - k_j - i + j)}{\prod_{0\leq i\leq d-1}(r + k_i - i)!} \prod_{0\leq i\leq d-1} s_{k_i}(\mathcal{E})$$

in $A^*(X) \otimes \mathbb{Q}$, where $k = (k_0, \dots, k_{d-1}) \in \mathbb{Z}^d_{\geq 0}$, and $s_i(\mathcal{E})$ is the i-th Segre class of $\mathcal{E}$.

The Segre classes $s_i(\mathcal{E})$ here are the ones satisfying $s(\mathcal{E},t)c(\mathcal{E},-t) = 1$ as in [2], [7], [8], where $s(\mathcal{E},t)$ and $c(\mathcal{E},t)$ are respectively the Segre series and the Chern polynomial of $\mathcal{E}$ in t. Note that our Segre class $s_i(\mathcal{E})$ differs by the sign $(-1)^i$ from the one in [3].

Theorem 0.1 yields

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Corollary 0.2 (Degree Formula for Grassmann Bundles)**.** *If X is projective and $\wedge^d \mathcal{E}$ is very ample, then $\mathbb{G}_X(d,\mathcal{E})$ is embedded in the projective space $\mathbb{P}(H^0(X,\wedge^d\mathcal{E}))$ by the tautological line bundle $\mathcal{O}_{\mathbb{G}_X(d,\mathcal{E})}(1)$, and its degree is given by*

$$\deg \mathbb{G}_X(d,\mathcal{E}) = (d(r-d)+n)! \sum_{|k|=n} \frac{\prod_{0\le i<j\le d-1}(k_i-k_j-i+j)}{\prod_{0\le i\le d-1}(r+k_i-i)!} \int_X \prod_{0\le i\le d-1} s_{k_i}(\mathcal{E}),$$

where $|k| := \sum_i k_i$.

Here a vector bundle $\mathcal{F}$ over X is said to be *very ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{F})}(1)$ of $\mathbb{P}_X(\mathcal{F})$ is very ample.

We also give a proof for the following:

Theorem 0.3 ([5], [10])**.** *We have*

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_\lambda \frac{1}{|\lambda+\varepsilon|!} f^{\lambda+\varepsilon} \Delta_\lambda(s(\mathcal{E}))$$

in $A^*(X)\otimes\mathbb{Q}$, where $\lambda = (\lambda_1,\dots,\lambda_d)$ is a partition with $|\lambda| := \sum_i \lambda_i$, $\varepsilon := (r-d)^d = (r-d,\dots,r-d)$, $f^{\lambda+\varepsilon}$ is the number of standard Young tableaux with shape $\lambda+\varepsilon$, and $\Delta_\lambda(s(\mathcal{E})) := \det[s_{\lambda_i+j-i}(\mathcal{E})]_{1\le i,j\le d}$ is the Schur polynomial in the Segre classes of $\mathcal{E}$ corresponding to λ.

Note that our proofs for Theorem 0.3 as well as Theorem 0.1 do not use the push-forward formula of Józefiak-Lascoux-Pragacz [6], while the proofs given in [5], [10] do. We establish instead a new push-forward formula, as follows: Let $\mathbb{F}^d_X(\mathcal{E})$ be the partial flag bundle of $\mathcal{E}$ on X, parametrizing flags of subbundles of corank 1 up to d in $\mathcal{E}$, let $p : \mathbb{F}^d_X(\mathcal{E}) \to X$ be the projection, and denote by $p_* : A^{*+c}(\mathbb{F}^d_X(\mathcal{E})) \to A^*(X)$ the push-forward by p, where c is the relative dimension of $\mathbb{F}^d_X(\mathcal{E})/X$. Let $\xi_0,\dots,\xi_{d-1}$ be the set of Chern roots of $\mathcal{Q}$. It turns out (see §1) that one may consider $A^{*+c}(\mathbb{F}^d_X(\mathcal{E}))$ as an $A^*(X)$-algebra generated by the ξ_i. Then

Theorem 0.4 (Push-Forward Formula)**.** *For any polynomial $F \in A^*(X)[T_0\dots,T_{d-1}]$, we have*

$$p_*F(\underline{\xi}) = \operatorname{const}_{\underline{t}} \Big(\Delta(\underline{t}) \prod_{i=0}^{d-1} t_i^{r-d} F(1/\underline{t}) \prod_{i=0}^{d-1} s(\mathcal{E},t_i) \Big),$$

in $A^*(X)$, where $\underline{\xi} := (\xi_0,\dots,\xi_{d-1})$, $\operatorname{const}_{\underline{t}}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\underline{t} := (t_0,\dots,t_{d-1})$, $\Delta(\underline{t}) := \prod_{0\le i<j\le d-1}(t_i-t_j)$ and $F(1/\underline{t}) := F(1/t_0,\dots,1/t_{d-1})$.

The contents of this article are organized as follows: The general theories [8, §6], [11, §§0–1] on the structure of Chow ring of certain partial flag bundles are reviewed in §1. Then, Theorem 0.4 is proved in §2, by which it is shown that $\pi_* \operatorname{ch}(\det Q)$ is given as the constant term of a certain Laurent series with coefficients in the Chow ring $A^*(X)$ of X, denoted by $P(\underline{t})$ (Proposition 2.5). To evaluate the constant term of $P(\underline{t})$, in §3, a linear form on the Laurent polynomial ring, denoted by Φ, is introduced (Definition 3.1), and an evaluation formula is proved (Proposition 3.3): The evaluation formula is the key in the final step to prove Theorems 0.1 and 0.3. In §4, a generalization of Cauchy determinant formula is given (Proposition 4.1). This yields another proof of a push-forward formula for monomials of the ξ_i (Lemma 2.2).

1. Set-up

Let X be a non-singular quasi-projective variety of dimension n defined over a field k, let $\mathcal{E}$ be a vector bundle of rank r on X, and let $\varpi : \mathbb{P}(\mathcal{E}) \to X$ be the projection. Denote by ξ the first Chern class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and define a polynomial $P_{\mathcal{E}} \in A^*(X)[T]$ associated to $\mathcal{E}$ by setting

$$P_{\mathcal{E}}(T) := T^r - c_1(\mathcal{E})T^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}),$$

where $A^*(X)$ is the Chow ring of X. Then, $P_{\mathcal{E}}(\xi) = 0$ by definition of the Chern classes ([3, Remark 3.2.4]), and

$$A^*(\mathbb{P}(\mathcal{E})) = \bigoplus_{0\le i\le r-1} A^*(X)\xi^i \simeq A^*(X)[T]/(P_{\mathcal{E}}(T)) \tag{1.1}$$

([3, Theorem 3.3 (b); Example 8.3.4]). Let $\varpi_* : A^{*+r-1}(\mathbb{P}(\mathcal{E})) \to A^*(X)$ be the push-forward by ϖ. Then $\varpi_*\alpha$ is equal to the coefficient of α in ξ^{r-1}, denoted by $\mathrm{coeff}_\xi(\alpha)$, with respect to the decomposition (1.1) for $\alpha \in A^{*+r-1}(\mathbb{P}(\mathcal{E}))$ ([3, Proposition 3.1]):

$$\varpi_*\alpha = \mathrm{coeff}_\xi(\alpha) \tag{1.2}$$

Denote by $\mathbb{F}^d_X(\mathcal{E})$ the partial flag bundle of $\mathcal{E}$ on X, parametrizing flags of subbundles of corank 1 up to d in $\mathcal{E}$, and let $p : \mathbb{F}^d_X(\mathcal{E}) \to X$ be the projection. Set $\mathcal{E}_0 := \mathcal{E}$, and let $\mathcal{E}_{i+1}$ be the kernel of the canonical surjection from the pull-back of $\mathcal{E}_i$ to $\mathbb{P}(\mathcal{E}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)$, with $\mathrm{rk}\,\mathcal{E}_i = r - i$ $(i \ge 0)$. Set $\xi_i := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1))$. We have an exact sequence on $\mathbb{P}(\mathcal{E}_i)$,

$$0 \to \mathcal{E}_{i+1} \to \mathcal{E}_i \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1) \to 0,$$

and an equation of Chern polynomials,

$$c(\mathcal{E}_i, t) = c(\mathcal{E}_{i+1}, t)(1 + \xi_i t), \tag{1.3}$$

where we omit the symbol of the pull-back by the projection $\mathbb{P}_{\mathbb{P}(\mathcal{E}_i)}(\mathcal{E}_{i+1}) \to \mathbb{P}(\mathcal{E}_i)$. It is easily shown that the projection $p : \mathbb{F}^d_X(\mathcal{E}) \to X$ decomposes as a successive composition of projective space bundles, $\mathbb{P}_{\mathbb{P}(\mathcal{E}_i)}(\mathcal{E}_{i+1}) \to \mathbb{P}(\mathcal{E}_i)$ $(i \ge 0)$:

$$p : \mathbb{F}^d_X(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{d-1}) \to \mathbb{P}(\mathcal{E}_{d-2}) \to \cdots \to \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_0) \to X.$$

In fact, $\mathbb{P}(\mathcal{E}_i) \simeq \mathbb{F}^{i+1}_X(\mathcal{E})$ $(0 \le i \le d-1)$. Using (1.1) repeatedly, we see that the Chow ring of $\mathbb{F}^d_X(\mathcal{E})$ is given as follows:

$$A^*(\mathbb{F}^d_X(\mathcal{E})) = \bigoplus_{\substack{0\le i_l\le r-l-1\\(0\le l\le d-1)}} A^*(X)\xi_0^{i_0}\xi_1^{i_1}\cdots\xi_{d-1}^{i_{d-1}} = \frac{A^*(X)[T_0, T_1, \ldots, T_{d-1}]}{(\{P_{\mathcal{E}_i}(T_i) | 0 \le i \le d-1\})}. \tag{1.4}$$

Denote by $p_* : A^{*+c}(\mathbb{F}^d_X(\mathcal{E})) \to A^*(X)$ the push-forward by p, where $c := \sum_{0\le i\le d-1}(r - i - 1)$, the relative dimension of $\mathbb{F}^d_X(\mathcal{E})/X$. Then, using (1.2) repeatedly, we see that

$$p_*\alpha = \mathrm{coeff}_{\underline{\xi}}(\alpha) \tag{1.5}$$

for $\alpha \in A^*(\mathbb{F}^d_X(\mathcal{E}))$, where $\mathrm{coeff}_{\underline{\xi}}(\alpha)$ denotes the coefficient of α in $\xi_0^{r-1}\xi_1^{r-2}\cdots\xi_{d-1}^{r-d}$ with respect to the decomposition (1.4).

Let $G := \mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of $\mathcal{E}$ on X, and let $\mathcal{Q} \leftarrow \pi^*\mathcal{E}$ be the universal quotient bundle of rank d. Consider the flag bundle $\mathbb{F}^{d-1}_G(\mathcal{Q})$ of $\mathcal{Q}$ on G, parametrizing flags of subbundles of corank 1 up to $d-1$ in $\mathcal{Q}$. Then, as in the case of $\mathbb{F}^d_X(\mathcal{E})$, the projection $\mathbb{F}^{d-1}_G(\mathcal{Q}) \to G$ decomposes as a successive composition of projective space bundles, $\mathbb{P}_{\mathbb{P}(\mathcal{Q}_i)}(\mathcal{Q}_{i+1}) \to \mathbb{P}(\mathcal{Q}_i)$ $(i \ge 0)$:

$$q : \mathbb{F}^{d-1}_G(\mathcal{Q}) = \mathbb{P}(\mathcal{Q}_{d-2}) \to \mathbb{P}(\mathcal{Q}_{d-2}) \to \cdots \to \mathbb{P}(\mathcal{Q}_1) \to \mathbb{P}(\mathcal{Q}_0) \to G,$$

where $\mathcal{Q}_0 := \mathcal{Q}$, and $\mathcal{Q}_{i+1}$ is the kernel of the canonical surjection from the pull-back of $\mathcal{Q}_i$ to $\mathbb{P}(\mathcal{Q}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1)$, with $\operatorname{rk}\mathcal{Q}_i = d-i$ $(i \geq 0)$: In fact, $\mathbb{P}(\mathcal{Q}_i) \simeq \mathbb{F}^{i+1}_G(\mathcal{Q})$ $(0 \leq i \leq d-2)$ and $\mathbb{P}_{\mathbb{P}(\mathcal{Q}_{d-2})}(\mathcal{Q}_{d-1}) \simeq \mathbb{P}(\mathcal{Q}_{d-2}) \simeq \mathbb{F}^{d-1}_G(\mathcal{Q}) = \mathbb{F}^d_G(\mathcal{Q})$. It follows from the construction of the $\mathcal{Q}_i$ that the Plücker class $\theta := c_1(\det\mathcal{Q}) = c_1(\mathcal{Q})$ is equal to the sum of the first Chern classes $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1))$ $(0 \leq i \leq d-1)$ in $A^*(\mathbb{F}^{d-1}_G(\mathcal{Q}))$, where $\mathcal{O}_{\mathbb{P}(\mathcal{Q}_{d-1})}(1) = \mathcal{Q}_{d-1}$ via $\mathbb{P}_{\mathbb{P}(\mathcal{Q}_{d-2})}(\mathcal{Q}_{d-1}) \simeq \mathbb{P}(\mathcal{Q}_{d-2})$.

It follows from the construction of the $\mathcal{E}_i$ that $\mathcal{E}_d$ is a corank d subbundle of $p^*\mathcal{E}$ on $\mathbb{F}^d_X(\mathcal{E})$, which induces a morphism, $r : \mathbb{F}^d_X(\mathcal{E}) \to G$ over X by the universal property of the Grassmann bundle G. Then it turns out that $\mathbb{F}^{d-1}_G(\mathcal{Q})$ is naturally isomorphic to $\mathbb{F}^d_X(\mathcal{E})$ over G via r, as is easily verified by using the universal property of flag bundles: We identify them via the natural isomorphism $\mathbb{F}^{d-1}_G(\mathcal{Q}) \simeq \mathbb{F}^d_X(\mathcal{E})$. Under this identification, it follows that $p = \pi \circ q$ and $\xi_i = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)) = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1))$ in $A^*(\mathbb{F}^d_X(\mathcal{E})) = A^*(\mathbb{F}^{d-1}_G(\mathcal{Q}))$ $(0 \leq i \leq d-1)$, where the symbol of pull-back to $\mathbb{F}^d_X(\mathcal{E}) = \mathbb{F}^{d-1}_G(\mathcal{Q})$ is omitted, as before. Thus we have

$$q^*\theta = \xi_0 + \cdots + \xi_{d-1} \tag{1.6}$$

in $A^*(\mathbb{F}^d_X(\mathcal{E})) = A^*(\mathbb{F}^{d-1}_G(\mathcal{Q}))$. For details, we refer to [8, §6], [11, §§0–1].

2. Laurent series

We keep the same notation as in §1.

Lemma 2.1. *For any non-negative integer p,*

$$\operatorname{coeff}_\xi(\xi^p) = \operatorname{const}_t(t^{-p+r-1}s(\mathcal{E}, t)),$$

where $\operatorname{const}_t(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in t.

Proof. Set $R_p(x_p, \dots, x_{p-r}) := \sum_{i=0}^r (-1)^i c_i(\mathcal{E}) x_{p-i}$, and consider a recurring relation, $R_p(x_p, \dots, x_{p-r}) = 0$ $(p \geq r)$ for $\{x_i\} \subseteq A^*(X)$. If $a_p := \operatorname{coeff}_\xi(\xi^p)$, then

$$R_p(a_p, \dots, a_{p-r}) = \operatorname{coeff}_\xi\left(\sum_{i=0}^r (-1)^i c_i(\mathcal{E})\xi^{p-i}\right) = 0$$

by $P_\mathcal{E}(\xi) = 0$. On the other hand, if $b_p := \operatorname{const}_t(t^{-p-1+r}s(\mathcal{E}, t))$, then

$$R_p(b_p, \dots, b_{p-r}) = \operatorname{const}_t\left(\sum_{i=0}^r c_i(\mathcal{E})(-t)^i t^{-p-1+r} s(\mathcal{E}, t)\right) = \operatorname{const}_t(t^{-p-1+r}) = 0$$

by $c(\mathcal{E}, -t)s(\mathcal{E}, t) = 1$. Thus both of $\{a_p\}$ and $\{b_p\}$ satisfy the recurring relation $R_p = 0$, so that $a_p = b_p$ for all p: Indeed, $a_r = b_r = c_1(\mathcal{E})$, $a_{r-1} = b_{r-1} = 1$ and $a_p = b_p = 0$ if $0 \leq p \leq r-2$. We here note that x_p is determined by $x_{p-1}, \dots, x_{p-r}$ if $R_p(x_p, \dots x_{p-r}) = 0$. □

Lemma 2.2. *For any non-negative integers $p_0, \dots, p_{d-1}$, we have*

$$\operatorname{coeff}_{\underline{\xi}}(\xi_0^{p_0} \cdots \xi_{d-1}^{p_{d-1}}) = \operatorname{const}_{\underline{t}}\left(\Delta(\underline{t}) \prod_{i=0}^{d-1} t_i^{-p_i+r-d} s(\mathcal{E}, t_i)\right),$$

where $\operatorname{const}_{\underline{t}}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\underline{t} := (t_0, \dots, t_{d-1})$, and $\Delta(\underline{t}) := \prod_{0 \leq i < j \leq d-1}(t_i - t_j)$ is the Vandermonde polynomial of $\underline{t}$.

Proof. Since $s(\mathcal{E}_{d-1}, t_{d-1}) = (1 - \xi_{d-2}t_{d-1})s(\mathcal{E}_{d-2}, t_{d-1})$ by (1.3), it follows from Lemma 2.1 that

$$\operatorname{coeff}_{\xi_{d-1}}(\xi_{d-1}^{p_{d-1}}) = \operatorname{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}(1 - \xi_{d-2}t_{d-1})s(\mathcal{E}_{d-2}, t_{d-1}))$$

in $A^*(\mathbb{P}(\mathcal{E}_{d-2}))$, where $\mathrm{coeff}_{\xi_{d-1}}(\cdots)$ denotes the coefficient of $\cdots$ in ξ_{d-1}^{r-d}. Therefore, using Lemma 2.1 again, we have

$$\begin{aligned}
&\mathrm{coeff}_{\xi_{d-2},\xi_{d-1}}(\xi_{d-2}^{p_{d-2}}\xi_{d-1}^{p_{d-1}})\\
&\quad=\mathrm{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_{d-2}}\,\mathrm{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}(1-\xi_{d-2}t_{d-1})s(\mathcal{E}_{d-2},t_{d-1})))\\
&\quad=\mathrm{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_{d-2}}\,\mathrm{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}s(\mathcal{E}_{d-2},t_{d-1})))\\
&\qquad+\mathrm{coeff}_{\xi_{d-2}}(\xi_{d-2}^{p_{d-2}+1}\,\mathrm{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}(-t_{d-1})s(\mathcal{E}_{d-2},t_{d-1})))\\
&\quad=\mathrm{const}_{t_{d-2}}(t_{d-2}^{-p_{d-2}+r-d+1}s(\mathcal{E}_{d-2},t_{d-2}))\,\mathrm{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}s(\mathcal{E}_{d-2},t_{d-1}))\\
&\qquad+\mathrm{const}_{t_{d-2}}(t_{d-2}^{-(p_{d-2}+1)+r-d+1}s(\mathcal{E}_{d-2},t_{d-2}))\,\mathrm{const}_{t_{d-1}}(t_{d-1}^{-p_{d-1}+r-d}(-t_{d-1})s(\mathcal{E}_{d-2},t_{d-1}))\\
&\quad=\mathrm{const}_{t_{d-2},t_{d-2}}(t_{d-2}^{-p_{d-2}+r-d+1}s(\mathcal{E}_{d-2},t_{d-2})t_{d-1}^{-p_{d-1}+r-d}s(\mathcal{E}_{d-2},t_{d-1}))\\
&\qquad+\mathrm{const}_{t_{d-2},t_{d-2}}(t_{d-2}^{-p_{d-2}+r-d}s(\mathcal{E}_{d-2},t_{d-2})t_{d-1}^{-p_{d-1}+r-d}(-t_{d-1})s(\mathcal{E}_{d-2},t_{d-1}))\\
&\quad=\mathrm{const}_{t_{d-2},t_{d-2}}\left((t_{d-2}-t_{d-1})\prod_{i=d-2}^{d-1}t_i^{-p_i+r-d}s(\mathcal{E}_{d-2},t_i)\right)
\end{aligned}$$

in $A^*(\mathbb{P}(\mathcal{E}_{d-3}))$, where $\mathrm{coeff}_{\xi_{d-2},\xi_{d-1}}(\cdots)$ denotes the coefficient of $\cdots$ in $\xi_{d-2}^{r-d+1}\xi_{d-1}^{r-d}$, and $\mathrm{coeff}_{\xi_{d-2}}(\cdots)$ the coefficient of $\cdots$ in ξ_{d-2}^{r-d+1}. Repeating this procedure, we obtain the conclusion. □

Remark 2.3. Expanding the determinant $\Delta(\underline{t})$ in the right-hand side in Lemma 2.2, using (1.5), we obtain a formula, $p_*(\xi_0^{p_0}\cdots\xi_{d-1}^{p_{d-1}})=\det[s_{p_i+j-r+1}(\mathcal{E})]_{0\le i,j\le d-1}$ in terms of the Schur polynomials in Segre classes of $\mathcal{E}$, which is equivalent to the determinantal formula [7, 8.1 Theorem] with $f_i(\xi_i):=\xi_i^{p_i}$ $(0\le i\le d-1)$.

Proposition 2.4. *For any polynomial $F\in A^*(X)[T_0,\dots,T_{d-1}]$, we have*

$$\mathrm{coeff}_{\underline{\xi}}(F(\underline{\xi}))=\mathrm{const}_{\underline{t}}\left(\Delta(\underline{t})\prod_{i=0}^{d-1}t_i^{r-d}F(1/\underline{t})\prod_{i=0}^{d-1}s(\mathcal{E},t_i)\right),$$

where $\underline{\xi}:=(\xi_0,\dots,\xi_{d-1})$, $\mathrm{const}_{\underline{t}}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\underline{t}:=(t_0,\dots,t_{d-1})$, $\Delta(\underline{t}):=\prod_{0\le i<j\le d-1}(t_i-t_j)$, and $F(1/\underline{t}):=F(1/t_0,\dots,1/t_{d-1})$.

Proof. This follows from Lemma 2.2. □

Proof of Theorem 0.4. The assertion follows from (1.5) and Proposition 2.4. □

Proposition 2.5. *With the same notation as in §1, we have*

$$\pi_*\,\mathrm{ch}(\det\mathcal{Q})=\mathrm{const}_{\underline{t}}(P(\underline{t})),$$

where $\pi_*:A^{*+d(r-d)}(\mathbb{G}_X(d,\mathcal{E}))\otimes\mathbb{Q}\to A^*(X)\otimes\mathbb{Q}$ is the push-forward by π, $\mathrm{ch}(\det\mathcal{Q})$ is the Chern character of $\det\mathcal{Q}$, $\mathrm{const}_{\underline{t}}(\cdots)$ denotes the constant term in the Laurent expansion of $\cdots$ in $\underline{t}:=(t_0,\dots,t_{d-1})$, and

$$P(\underline{t}):=\Delta(\underline{t})\prod_{i=0}^{d-1}t_i^{r-d-(d-1-i)}\exp\left(\sum_{i=0}^{d-1}\frac{1}{t_i}\right)\prod_{i=0}^{d-1}s(\mathcal{E},t_i).$$

Note that, though $\exp\left(\sum_{i=0}^{d-1}\frac{1}{t_i}\right)$ is an element in $\mathbb{Q}[[\{\frac{1}{t_i}\}_{0\le i\le d-1}]]$, $\mathrm{const}_{\underline{t}}(P(\underline{t}))$ is well defined since the Segre series are polynomials in $\underline{t}$.

Proof. Since $\mathbb{F}_G^{i+1}(\mathcal{Q}) \to \mathbb{F}_G^i(\mathcal{Q})$ is a $\mathbb{P}^{d-1-i}$-bundle, using [3, Proposition 3.1] repeatedly, for a non-negative integer N, we have

$$\theta^N = q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}q^*\theta^N),$$

where q is the composition of the projections, $\mathbb{F}_G^{d-1}(\mathcal{Q}) \to \cdots \to \mathbb{F}_G^1(\mathcal{Q}) \to G$. It follows from (1.6) and the commutativity $p = \pi \circ q$ via the identification $\mathbb{F}_G^{d-1}(\mathcal{Q}) = \mathbb{F}_X^d(\mathcal{E})$ that

$$\begin{aligned}\pi_*(\theta^N) =& \pi_* q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}q^*\theta^N)\\ =& \pi_* q_*\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N\Big) = p_*\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N\Big),\end{aligned}$$

where p is the composition of the projections, $\mathbb{F}_X^d(\mathcal{E}) \to \cdots \to \mathbb{F}_X^1(\mathcal{E}) \to X$. Now, apply Theorem 0.4 with $F := \prod_{i=0}^{d-1} T_i^{d-1-i}\Big(\sum_{i=0}^{d-1} T_i\Big)^N$. Then,

$$p_*\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N\Big) = \mathrm{const}_{\underline{t}}\Big(\Delta(\underline{t})\prod_{i=0}^{d-1} t_i^{r-d-(d-1-i)}\Big(\sum_{i=0}^{d-1} t_i^{-1}\Big)^N\prod_{i=0}^{d-1} s(\mathcal{E}, t_i)\Big).$$

Thus the conclusion follows with $\mathrm{ch}(\det\mathcal{Q}) = \exp(\theta)$. □

3. A linear form on the Laurent polynomial ring

Definition 3.1. Let A be a $\mathbb{Q}$-algebra. We define a linear form $\Phi : A[\{t_i, \frac{1}{t_i}\}_{0\le i\le d-1}] \to A$ on the Laurent polynomial ring $A[\{t_i, \frac{1}{t_i}\}_{0\le i\le d-1}]$ by

$$\Phi(f) := \mathrm{const}_{\underline{t}}\Big(\Delta(\underline{t})\exp\Big(\sum_{i=0}^{d-1}\frac{1}{t_i}\Big)f(\underline{t})\Big) \qquad \Big(f \in A\Big[\Big\{t_i, \frac{1}{t_i}\Big\}_{0\le i\le d-1}\Big]\Big),$$

where $\underline{t} := (t_0, \ldots, t_{d-1})$.

Lemma 3.2. (1) *Consider the natural action of the permutation group $\mathfrak{S}_d$ on $A[\{t_i, \frac{1}{t_i}\}_{0\le i\le d-1}]$ with $\sigma(t_i) := t_{\sigma(i)}$ $(\sigma \in \mathfrak{S}_d)$. Then we have $\Phi(\sigma(f)) = \mathrm{sgn}(\sigma)\Phi(f)$. As a consequence, we have*

$$\Phi\Big(\prod_{i=0}^{d-1} t_i^{-(d-1-i)}f(\underline{t})\Big) = (-1)^{d(d-1)/2}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-i}f(\underline{t})\Big)$$

for a symmetric function $f(\underline{t})$.

(2) *For a Schur polynomial $s_\lambda(\underline{t})$ and a symmetric function $f(\underline{t})$, we have*

$$\Phi\Big(\prod_{i=0}^{d-1} t_i^{-i}f(\underline{t})s_\lambda(\underline{t})\Big) = \Phi\Big(\prod_{i=0}^{d-1} t_i^{-i+\lambda_{i+1}}f(\underline{t})\Big).$$

Here the *Schur polynomial* $s_\lambda(\underline{t})$ in $\underline{t} = (t_0, \ldots, t_{d-1})$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ is the polynomial defined by

$$s_\lambda(\underline{t}) := \frac{\det[t_j^{\lambda_i+d-i}]}{\det[t_j^{d-i}]} = \frac{\det[t_j^{\lambda_i+d-i}]}{\Delta(\underline{t})},$$

where $1 \le i \le d$, $0 \le j \le d-1$ (see, *e.g.*, [3, 14.5 and A.9], [9, Chapter I, §3]).

Proof. (1). The assertion is a direct consequence from the definition of Φ and a property of $\Delta(\underline{t})$.

(2). Using (1), we have

$$\begin{aligned}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-i} f(\underline{t}) s_\lambda(\underline{t})\Big) &= \frac{1}{d!}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-(d-1)} f(\underline{t}) s_\lambda(\underline{t}) \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma) \prod_{i=0}^{d-1} t_{\sigma(i)}^{d-1-i}\Big)\\
&= \frac{1}{d!}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-(d-1)} f(\underline{t}) s_\lambda(\underline{t}) \Delta(\underline{t})\Big)\\
&= \frac{1}{d!}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-(d-1)} f(\underline{t}) \det[t_j^{\lambda_l+d-l}]_{1\le l\le d, 0\le j\le d-1}\Big)\\
&= \frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma)\Phi\Big(\prod_{i=0}^{d-1} t_{\sigma(i)}^{-i+\lambda_{i+1}} f(\underline{t})\Big) = \Phi\Big(\prod_{i=0}^{d-1} t_i^{-i+\lambda_{i+1}} f(\underline{t})\Big). \qquad \square\end{aligned}$$

To simplify the notation, for a finite set of integers $\{a_i\}_{0\le i\le d-1}$, set

$$\{a_i\}! := \prod_{0\le i\le d-1} a_i!, \quad \Delta(a_i) := \prod_{0\le i<j\le d-1} (a_i - a_j).$$

Setting $m! := \Gamma(m+1)$ for $m \in \mathbb{Z}$, we have $1/m! = 0$ if $m < 0$.

Proposition 3.3 (Evaluation Formula)**.** *For $k = (k_0, \dots, k_{d-1}) \in \mathbb{Z}^d_{\ge 0}$, we have*

$$\Phi\Big(\prod_{i=0}^{d-1} t_i^{k_i}\Big) = \frac{(-1)^{d(d-1)/2}\Delta(k_i)}{\{k_i + d - 1\}!}.$$

Proof. We have

$$\begin{aligned}\Phi\Big(\prod_{i=0}^{d-1} t_i^{k_i}\Big) &= \operatorname{const}_{\underline{t}}\Big(\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma)\prod_{i=0}^{d-1}\Big(t_i^{k_i+d-1-\sigma(i)}\exp\Big(\frac{1}{t_i}\Big)\Big)\Big)\\
&= \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma)\prod_{i=0}^{d-1} \operatorname{const}_{t_i}\Big(t_i^{k_i+d-1-\sigma(i)}\exp\Big(\frac{1}{t_i}\Big)\Big)\\
&= \sum_{\sigma\in\mathfrak{S}_d} \frac{\operatorname{sgn}(\sigma)}{\{k_i+d-1-\sigma(i)\}!} = \det\left[\frac{1}{(k_i+d-1-j)!}\right]_{0\le i,j\le d-1}\\
&= \frac{(-1)^{d(d-1)/2}\Delta(k_i)}{\{k_i+d-1\}!}.\end{aligned}$$

The last equality follows from the lemma below. $\square$

Lemma 3.4 ([3, Example A.9.3])**.**

$$\det\left[\frac{1}{(x_i+j)!}\right]_{0\le i,j\le d-1} = \frac{\Delta(x_i)}{\{x_i+d-1\}!}.$$

Proof of Theorem 0.1. By Proposition 2.5 and Lemma 3.2 (1) with $A := A^*(X)\otimes\mathbb{Q}$, we have

$$\begin{aligned}\text{(3.1)}\quad \pi_* \operatorname{ch}(\det \mathcal{Q}) &= \Phi\Big(\prod_{i=0}^{d-1} t_i^{-(d-1-i)} \prod_{i=0}^{d-1}\big(t_i^{r-d} s(\mathcal{E}, t_i)\big)\Big)\\
&= (-1)^{d(d-1)/2}\Phi\Big(\prod_{i=0}^{d-1} t_i^{-i}\prod_{i=0}^{d-1}\big(t_i^{r-d} s(\mathcal{E}, t_i)\big)\Big).\end{aligned}$$

Since

$$\prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \sum_k \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E}) t_i^{k_i},$$

it follows from Proposition 3.3 that the most right-hand side of (3.1) is equal to

$$(-1)^{d(d-1)/2} \sum_k \Phi\Big(\prod_{i=0}^{d-1} t_i^{r-d+k_i-i} \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E})\Big) = \sum_k \frac{\Delta(k_i - i)}{\{r + k_i - i - 1\}!} \prod_{i=0}^{d-1} s_{k_i}(\mathcal{E}),$$

where $k = (k_0, \ldots, k_{d-1}) \in \mathbb{Z}^d_{\geq 0}$. Thus we obtain the conclusion. □

Proof of Corollary 0.2. By the assumption $\mathbb{G}_X(d, \mathcal{E})$ is projective and the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\wedge^d \mathcal{E})}(1)$ defines an embedding $\mathbb{P}_X(\wedge^d \mathcal{E}) \hookrightarrow \mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$. Therefore $\mathbb{G}_X(d, \mathcal{E})$ is considered to be a projective variety in $\mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$ via the relative Plücker embedding $\mathbb{G}_X(d, \mathcal{E}) \hookrightarrow \mathbb{P}_X(\wedge^d \mathcal{E})$ over X defined by the quotient $\wedge^d \pi^* \mathcal{E} \to \wedge^d \mathcal{Q} = \det \mathcal{Q}$. Since the hyperplane section class of $\mathbb{G}_X(d, \mathcal{E})$ is equal to the Plücker class θ, we obtain the conclusion, taking the degree of the equality in Theorem 0.1. □

Proof of Theorem 0.3. By Lemmas 3.5 below, 3.2 (2) and Proposition 3.3, the most right-hand side of (3.1) is equal to

$$\begin{aligned}
(-1)^{d(d-1)/2} &\sum_\lambda \Phi\Big(\prod_{i=0}^{d-1} t_i^{r-d-i} s_\lambda(\underline{t})\Big) \Delta_\lambda(s(\mathcal{E})) \\
&= (-1)^{d(d-1)/2} \sum_\lambda \Phi\Big(\prod_{i=0}^{d-1} t_i^{r-d-i+\lambda_{i+1}}\Big) \Delta_\lambda(s(\mathcal{E})) \\
&= \sum_\lambda \frac{\Delta(r - d - i + \lambda_{i+1})}{\{r - d - i + \lambda_{i+1} + (d-1)\}!} \Delta_\lambda(s(\mathcal{E})) \\
&= \sum_\lambda \frac{\Delta(\lambda_{i+1} - (i+1))}{\{\lambda_{i+1} + r - (i+1)\}!} \Delta_\lambda(s(\mathcal{E})) = \sum_\lambda \frac{f^{\lambda+\varepsilon}}{|\lambda + \varepsilon|!} \Delta_\lambda(s(\mathcal{E})).
\end{aligned}$$

□

Lemma 3.5.

$$\prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \sum_\lambda \Delta_\lambda(s(\mathcal{E})) s_\lambda(\underline{t}).$$

Proof. Using Cauchy identity [9, Chapter I, (4.3)] and Jacobi-Trudi identity [3, Lemma A.9.3], we have

$$\prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \prod_{i=0}^{d-1} \frac{1}{c(\mathcal{E}, -t_i)} = \prod_{i=0}^{d-1} \prod_{j=1}^{r} \frac{1}{1 - \alpha_j t_i} = \sum_\lambda s_\lambda(\underline{\alpha}) s_\lambda(\underline{t}) = \sum_\lambda \Delta_\lambda(s(\mathcal{E})) s_\lambda(\underline{t}),$$

where $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_r\}$ are the Chern roots of the vector bundle $\mathcal{E}$. □

4. Appendix: A generalization of Cauchy Determinant Formula

Consider a polynomial ring $R_1 := A[\xi_0, \ldots, \xi_{r-1}]$ with r variables over a $\mathbb{Q}$-algebra A. Denote by c''_i the i-th elementary symmetric polynomial in $\xi_d, \ldots, \xi_{r-1}$, and by c_i the i-th elementary symmetric polynomial in $\xi_0, \ldots, \xi_{r-1}$. We define the Segre series $s(t)$ by

$$s(t) := \frac{1}{\prod_{i=0}^{r-1}(1 - \xi_i t)}.$$

Set $R_2 := A[\xi_0, \ldots, \xi_{d-1}, c''_1, \ldots, c''_{r-d}]$, and $R_3 := A[c_1, \ldots, c_r]$. Then, $R_1 \supset R_2 \supset R_3$, and R_1 (resp. R_2) is a free R_3-modules generated by $\{\xi_0^{i_0} \cdots \xi_{r-1}^{i_{r-1}}\}$ (resp. $\{\xi_0^{i_0} \cdots \xi_{d-1}^{i_{d-1}}\}$), where $0 \leq i_l \leq r-l-1$ (see, *e.g.*, [1, Chapitre 4, §6], [7, §§2–3]). In particular, we have a decomposition,

$$R_2 = \bigoplus_{\substack{0 \leq i_l \leq r-l-1 \\ (0 \leq l \leq d-1)}} R_3 \cdot \xi_0^{i_0} \xi_1^{i_1} \cdots \xi_{d-1}^{i_{d-1}}. \tag{4.1}$$

For $\alpha \in R_2$, we denote by $\operatorname{coeff}_{\underline{\xi}}(\alpha)$ the coefficient of α in $\xi_0^{r-1} \cdots \xi_{d-1}^{r-d}$ with respect to the decomposition (4.1).

Let $\mathcal{A}$ (resp. $\mathcal{A}'$, $\mathcal{A}''$) be the anti-symmetrizer for variables $\{\xi_0, \ldots, \xi_{r-1}\}$ (resp. $\{\xi_0, \ldots, \xi_{d-1}\}$, $\{\xi_d, \ldots, \xi_{r-1}\}$), that is, $\mathcal{A}(\alpha) := \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma)\sigma(\alpha)$ $(\alpha \in R_1)$, for instance.

Proposition 4.1 (Generalization of Cauchy Determinant Formula)**.** *We have an equality*

$$\mathcal{A}\Big(\frac{\Delta(\xi_0, \ldots, \xi_{d-1})\Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i,j \leq d-1}(\tau_j - \xi_i)}\Big) = \frac{\Delta(\xi_0, \ldots, \xi_{r-1})}{\prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1}(\tau_j - \xi_i)}.$$

By setting $\tau_i := \dfrac{1}{t_i}$, *we have*

$$\mathcal{A}\Big(\frac{\Delta(\xi_0, \ldots, \xi_{d-1}) \cdot \Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i,j \leq d-1}(1 - \xi_i t_j)}\Big) = \frac{\Delta(\xi_0, \ldots, \xi_{r-1}) \prod_{i=0}^{d-1} t_i^{r-d}}{\prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1}(1 - \xi_i t_j)}.$$

Proof. The fractional expression,

$$\mathcal{A}\Big(\frac{\Delta(\xi_0, \ldots, \xi_{d-1})\Delta(\xi_d, \ldots, \xi_{r-1})}{\prod_{0 \leq i,j \leq d-1}(\tau_j - \xi_i)}\Big) \prod_{0 \leq i \leq r-1, 0 \leq j \leq d-1} (\tau_j - \xi_i)$$

is actually a homogeneous polynomial in the variables, $\xi_0, \ldots, \xi_{r-1}$, $\tau_0, \ldots, \tau_{r-1}$, with degree $d(d-1)/2 + (r-d)(r-d-1)/2 - d^2 + rd = r(r-1)/2$, and anti-symmetric with respect to the ξ_i. Therefore it is a multiple of $\Delta(\xi_0, \ldots, \xi_{r-1})$. By comparing the coefficient of $\xi_0^{r-1} \cdots \xi_{r-1}^0$, we see that those polynomials are equal to each other, and we obtain the first equality. The second equality follows from the first one. □

Another Proof of Lemma 2.2. Let $G(\underline{t})$ be the generating function of $\operatorname{coeff}_{\underline{\xi}}(\xi_0^{p_0} \cdots \xi_{d-1}^{p_{d-1}})$, that is,

$$G(\underline{t}) := \sum_{p_0, \ldots, p_{d-1} \geq 0} \operatorname{coeff}_{\underline{\xi}}(\xi_0^{p_0} \cdots \xi_{d-1}^{p_{d-1}}) t_0^{p_0} \cdots t_{d-1}^{p_{d-1}}.$$

For $0 \leq i_l \leq r-l-1$, we have

$$\mathcal{A}(\xi_0^{i_0} \cdots \xi_{r-1}^{i_{r-1}}) = \begin{cases} \Delta(\xi_0, \ldots, \xi_{r-1}), & (i_0, \ldots, i_{r-1}) = (r-1, \ldots, 0), \\ 0, & (i_0, \ldots, i_{r-1}) \neq (r-1, \ldots, 0). \end{cases}$$

Since $\mathcal{A}$ is R_3-linear, we have an equality

$$\mathcal{A}(\alpha \cdot \xi_d^{r-d-1} \cdots \xi_{r-1}^0) = \operatorname{coeff}_{\underline{\xi}}(\alpha)\Delta(\xi_0, \ldots, \xi_{r-1})$$

in R_1 for $\alpha \in R_2$. Therefore,

$$\begin{aligned}
\Delta(\xi_0,\dots,\xi_{r-1})G(\underline{t}) &= \sum_{p_0,\dots,p_{d-1}\geq 0} \mathcal{A}(\xi_0^{p_0},\dots,\xi_{d-1}^{p_{d-1}}\cdot\xi_d^{r-d-1}\cdots\xi_{r-1}^0)\ t_0^{p_0}\cdots t_{d-1}^{p_{d-1}} \\
&= \mathcal{A}\Big(\frac{\xi_d^{r-d-1}\cdots\xi_{r-1}^0}{(1-\xi_0t_0)\cdots(1-\xi_{d-1}t_{d-1})}\Big) \\
&= \mathcal{A}\Big(\mathcal{A}'\Big(\frac{1}{(1-\xi_0t_0)\cdots(1-\xi_{d-1}t_{d-1})}\Big)\mathcal{A}''(\xi_d^{r-d-1}\cdots\xi_{r-1}^0)\Big) \\
&= \mathcal{A}\Big(\frac{\Delta(t_0,\dots,t_{d-1})\Delta(\xi_0,\dots,\xi_{d-1})\Delta(\xi_d,\dots,\xi_{r-1})}{\prod_{0\leq i,j\leq d-1}(1-\xi_it_j)}\Big) \\
&= \mathcal{A}\Big(\frac{\Delta(\xi_0,\dots,\xi_{d-1})\Delta(\xi_d,\dots,\xi_{r-1})}{\prod_{0\leq i,j\leq d-1}(1-\xi_it_j)}\Big)\Delta(t_0,\dots,t_{d-1}).
\end{aligned}$$

Here we used the equality,

$$\mathcal{A}(f(\xi_0,\dots,\xi_{d-1})g(\xi_d,\dots,\xi_{r-1})) = \mathcal{A}(\mathcal{A}'(f(\xi_0,\dots,\xi_{d-1}))\mathcal{A}''(g(\xi_d,\dots,\xi_{r-1})))$$

and Cauchy determinant formula ([9, p.67, I.4, Example 6]). Finally, using Proposition 4.1, we see that

$$G(\underline{t}) = \frac{\Delta(t_0,\dots,t_{d-1})\prod_{i=0}^{d-1}t_i^{r-d}}{\prod_{0\leq i\leq r-1,0\leq j\leq d-1}(1-\xi_it_j)} = \Delta(t_0,\dots,t_{d-1})\prod_{i=0}^{d-1}t_i^{r-d}s(t_i),$$

and this proves Lemma 2.2 with $R_1 := A^*(X)$ and $R_2 := A^*(\mathbb{F}_X^d(\mathcal{E})) = A^*(\mathbb{F}_G^{d-1}(\mathcal{Q}))$. $\square$

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